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Left regular representation of $sl_q(3)$: reduction and intertwiners

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Abstract. Reduction of the left regular representation of the quantum algebra $sl_q(3)$ is studied and q-difference intertwining operators are constructed. The irreducible representations correspond to a q-deformation of the spaces of local sections of certain line bundles over the flag manifold.

1. Introduction

Representations of quantum groups in terms of q-difference operators and their physical applications to coherent states and Fock-space representations have recently become the topic of intense study, see e.g. [1–4]. Furthermore, reduction of the left regular representation to an infinite family of (reducible and irreducible) representations and the q-difference intertwining operators have been studied in examples of Lorentz quantum algebra [5], $sl_q(2)$ and its contraction to $e_q(2)$ [6], and following the canonical procedure [7] for $q = 1$.

In this paper we investigate the case of $sl_q(3)$, a quantization of the simple Lie algebra of rank two, which besides being computationally more involved, presents some new features useful for a generalization to the case of $sl_q(n)$ and then to q-deformations of all semisimple Lie algebras. This will help us to understand the important relation between representation theory and geometry of quantum groups better.

We shall denote by G_q the quantum group $\mathrm{Fun}_q(G)$ and by g_q its dual $U_q(g)$, a quantization of the Lie algebra g of G. Recall that the left regular representation $L : g_q \times G_q \to G_q$, $(a, \varphi) \to \mathcal{L}(a)\varphi$, and the right regular representation $R : g_q \times G_q \to G_q$, $(a, \varphi) \to \mathcal{R}(a)\varphi$, of g_q are defined on G_q by

$$(\mathcal{L}(a)\varphi)\,(b) = \varphi(S(a)b) \tag{1.1}$$

$$(\mathcal{R}(a)\varphi)\,(b) = \varphi(ba) \tag{1.2}$$

respectively, where $a, b \in g_q$, $\varphi \in G_q$ and S is the antipode.

We shall reduce the left regular representation on the eigenspaces of the right regular representation of the Borel generators; namely, we shall impose the condition that the Cartan generators act as a multiplication by fixed numbers (to be specified further) and that the other Borel generators vanish. Some of the remaining generators of g then yield (via the right regular representation) interesting intertwining q-difference operators, the kernels of which provide a further reduction.

Symmetry considerations have been used as a very powerful method in the developement of theoretical physics in recent years. Several equations of mathematical physics are

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interesting merely for the reason that they possess an underlying symmetry. The differential operators corresponding to such equations can be viewed as intertwiners between some representations of this symmetry group. Our approach may yield a preferred deformation of such intertwiners and related equations which otherwise is highly non-unique. One expects the most interesting examples to be provided by non-simple groups and algebras (containing e.g. the translations), see e.g. [8], but their q-analogue is not yet fully developed. For that reason it is interesting to study the simple Lie algebras, the lower dimensional examples of which are certainly relevant for physical applications. In the simplest case $sl_q(2)$ [6] the intertwining operator turns out to be the widely known q-derivative; a q-deformation of the holomorphic (Dolbeault) derivative and a pair of such operators have been found in [5]. The first less well known case to which we devote our study is $sl_q(3)$ (or its real form $su_q(3)$). (The case of $sl_q(4)$, or its real form $su_q(2,2)$, is interesting for providing a q-conformally covariant deformation of the Maxwell equations but is, however, outside the scope of this paper.)

2. Preliminaries on $sl_q(3)$ and $SL_q(3)$

The generators of $SL_q(3)$ (quantization of the algebra of complex functions on $SL_q(3)$) are the unit 1 and t_{ij}, with $i, j = 1, 2, 3$, arranged as a 3×3 matrix T, satisfying the commutation relations

$$RT_1T_2 = T_2T_1R \qquad (T_1 = T \otimes I,\ T_2 = I \otimes T) \tag{2.1}$$

where

$$q^{\frac{1}{3}} R = q \sum_{i}^{3} e_{ii} \otimes e_{ii} + \sum_{i \neq j}^{3} e_{ii} \otimes e_{jj} + \lambda \sum_{i>j}^{3} e_{i,j} \otimes e_{j,i} \tag{2.2}$$

with $\lambda = q - q^{-1}$. More explicitly, we have

$$\begin{aligned} &t_{ij}t_{kl} = q t_{kl} t_{ij} \qquad i = k,\ j < l, \text{ or } i < k,\ j = l \\ &[t_{ij}, t_{kl}] = \lambda t_{il} t_{kj} \qquad i < k,\ j < l \\ &[t_{ij}, t_{kl}] = 0 \qquad i < k,\ j > l. \end{aligned} \tag{2.3}$$

An additional relation is

$$\det_q = t_{11}(t_{22}t_{33} - q t_{23}t_{32}) - q t_{21}(t_{12}t_{33} - q t_{13}t_{32}) + q^2 t_{31}(t_{12}t_{23} - q t_{13}t_{22}) = 1 \tag{2.4}$$

which can also be written in an equivalent form as

$$\det_q = t_{11}(t_{22}t_{33} - q t_{23}t_{32}) - q t_{12}(t_{21}t_{33} - q t_{23}t_{31}) + q^2 t_{13}(t_{21}t_{32} - q t_{22}t_{31}) = 1. \tag{2.5}$$

Considered as a Hopf algebra, $SL_q(3)$ has the co-multiplication Δ, co-unit ε, and antipode S given on the generators by

$$\Delta t_{ij} = t_{ik} \otimes t_{kj} \tag{2.6}$$

$$\varepsilon t_{ij} = \delta_{ij} \tag{2.7}$$

$$S t_{ij} = (-q)^{i-j} \tilde{t}_{ji} \tag{2.8}$$

where $\tilde{t}_{ij}$ are the quantum minors. For $\bar{q} = q$, with the $*$-conjugation (complex anti-linear algebra anti-involution and co-algebra involution) $t^*_{ij} = St_{ji}$, $SL_q(3)$ becomes a Hopf $*$-algebra denoted by $SU_q(3)$.

As generators of the algebra $sl_q(3)$, dual to $SL_q(3)$, we shall use the unit 1 and the functionals which have been introduced in [9]: l^+_{ij} with $i < j$, l^-_{ij} with $i > j$, l^+_{ii} and l^-_{ii} where $i, j = 1, 2, 3$. When arranged in upper- and lower-triangular matrices $L^\pm$ respectively, they are defined by the duality conditions

$$(L^\pm, T_1 \dots T_m) = R^\pm_1 \dots R^\pm_m \qquad \text{for } m = 1, 2, \dots \tag{2.9}$$

where for $1 \leqslant \ell \leqslant m$, $T^\pm_\ell$ act in the ℓth factor, $R^\pm_\ell$ act in the factors with indices 0 and ℓ of $(\mathbb{C}^3)^{\otimes(m+1)}$, $R^+ = PRP$, $R^- = R^{-1}$ and $P \in \mathrm{Mat}(\mathbb{C}^3 \otimes \mathbb{C}^3)$ is the permutation matrix.

The commutation relations are

$$R^+ L^\pm_1 L^\pm_2 = L^\pm_2 L^\pm_1 R^+ \qquad R^+ L^+_1 L^-_2 = L^-_2 L^+_1 R^+. \tag{2.10}$$

Additional relations are (see [9, 10])

$$l^+_{ii} l^-_{ii} = 1 \qquad i = 1, 2, 3 \tag{2.11}$$

$$l^+_{11} l^+_{22} l^+_{33} = l^-_{11} l^-_{22} l^-_{33} = 1. \tag{2.12}$$

The Hopf algebra structure is then given by

$$\Delta l^\pm_{ij} = l^\pm_{ik} \otimes l^\pm_{kj} \tag{2.13}$$

$$\varepsilon l^\pm_{ij} = \delta_{ij} \tag{2.14}$$

and the antipode which is defined as for T but with the replacement of q by q^{-1}. The $l^\pm_{ij}$ can be expressed in terms of the more popular generators q^{H_i}, $X^\pm_i$, $i = 1, 2$ (see [9, 11]) via

$$\begin{aligned} & q^{H_1} = l^+_{11}(l^+_{22})^{-1} \qquad q^{H_2} = l^+_{11}(l^+_{22})^2 \\ & X^+_1 = t^{-1} q^{-1/6} l^+_{12} (l^+_{11})^{-1/2} (l^+_{22})^{-1/2} \qquad X^+_2 = t^{-1} q^{-1/6} l^+_{23} (l^+_{11})^{1/2} \\ & X^-_1 = -t^{-1} q^{1/6} l^-_{21} (l^+_{11})^{1/2} (l^+_{22})^{1/2} \qquad X^-_2 = -t^{-1} q^{1/6} l^-_{32} (l^+_{11})^{-1/2}. \end{aligned} \tag{2.15}$$

They satisfy the commutation relations

$$\begin{aligned} & [H_i, H_j] = 0 \qquad [H_i, X^\pm_j] = \pm(\alpha_i, \alpha_j) X^\pm_j \\ & [X^+_i, X^-_j] = \delta_{ij} [H_i] \qquad i, j = 1, 2 \\ & \sum_{k=0}^{m} (-1)^k \binom{m}{k}_q q^{k(k-m)(\alpha_i,\alpha_i)/2} (X^\pm_i)^k X^\pm_j (X^\pm_i)^{m-k} = 0 \qquad \text{for } i \neq j \end{aligned} \tag{2.16}$$

where

$$[H_i] = \frac{(q^{H_i} - q^{-H_i})}{(q - q^{-1})} \qquad m = 1 - A_{ij}$$

$$\binom{m}{k}_q = \frac{(q^m - 1)(q^{m-1} - 1) \cdots (q^{m-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}.$$

The Hopf algebra structure is then given by

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \qquad \Delta(X^\pm_i) = X^\pm_i \otimes q^{-H_i/2} + q^{H_i/2} \otimes X^\pm_i \tag{2.17}$$

$$\varepsilon(X^\pm_i) = 0 \qquad \varepsilon(H_i) = 0 \tag{2.18}$$

$$S(X^\pm_i) = q^{-\rho} X^\pm_i q^\rho \qquad S(H_i) = -H_i \tag{2.19}$$

with $\rho = \sum H_\alpha/2$, where α belongs to the set of positive roots.

3. The left regular representation of $sl_q(3)$ on $SL_q(3)$

It can easily be seen from the definition (1.1) that the left regular representation of $l_{ij}^{\pm}$ amounts to a multiplication of the generator matrix T from the left by the following numerical matrices:

$$
\begin{aligned}
&(Sl_{11}^{+}, T) = q^{\frac{1}{3}} \begin{pmatrix} \frac{1}{q} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (Sl_{12}^{+}, T) = -q^{\frac{1}{3}}\lambda \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&(Sl_{13}^{+}, T) = -q^{\frac{1}{3}}\lambda \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad (Sl_{21}^{-}, T) = q^{-\frac{1}{3}}\lambda \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&(Sl_{22}^{+}, T) = q^{\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{q} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (Sl_{23}^{+}, T) = -q^{\frac{1}{3}}\lambda \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
&(Sl_{31}^{-}, T) = q^{-\frac{1}{3}}\lambda \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (Sl_{32}^{-}, T) = q^{-\frac{1}{3}}\lambda \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
&(Sl_{33}^{+}, T) = q^{\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{q} \end{pmatrix}.
\end{aligned}
\tag{3.1}
$$

Similarly, the right regular representation amounts to a multiplication of T from the right by the following matrices:

$$
\begin{aligned}
&(T, l_{11}^{+}) = q^{-\frac{1}{3}} \begin{pmatrix} q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (T, l_{12}^{+}) = q^{-\frac{1}{3}}\lambda \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&(T, l_{13}^{+}) = q^{-\frac{1}{3}}\lambda \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad (T, l_{21}^{-}) = -q^{\frac{1}{3}}\lambda \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&(T, l_{22}^{+}) = q^{-\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (T, l_{23}^{+}) = q^{-\frac{1}{3}}\lambda \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
&(T, l_{31}^{-}) = -q^{\frac{1}{3}}\lambda \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (T, l_{32}^{-}) = -q^{\frac{1}{3}}\lambda \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
&(T, l_{33}^{+}) = q^{-\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{pmatrix}.
\end{aligned}
\tag{3.2}
$$

As an (overcomplete) basis of $SL_q(3)$ we can take the ordered monomials

$$t^{\bar{n}} =: t_{11}^{n_{11}} t_{12}^{n_{12}} \dots t_{33}^{n_{33}} \tag{3.3}$$

where $\bar{n} =: (n_{11}, n_{12}, \dots, n_{33})$ with $n_{ij} \in \mathbb{Z}_+$ and we use (unless otherwise stated) the usual ordering, i.e. first according to the row index and then the column index.

In what follows we shall need an *independent* basis for $SL_q(3)$ which we believe must be known but which we have been unable to find in any literature for $SL_q(n)$ with $n \geqslant 3$. In order to obtain such an independent basis we have to restrict the elements $t^{\bar{n}}$ by the $\det_q = 1$ condition (2.2). We proceed as follows.

Assume for a moment a different ordering according to which the maximal power $(t_{11}t_{22}t_{33})^p$, where $p = \min\{n_{11}, n_{22}, n_{33}\}$, of the product of diagonal generators t_{11}, t_{22} and t_{33} comes first, then the remaining powers of (at most two of) t_{ii} and finally the other generators t_{ij}, $i \neq j$, in the usual order. (This is a convenient choice, inessential for the outcome.) Using (2.2), $t_{11}t_{22}t_{33}$ can be expressed as a polynomial in t_{ij} of degree of at most one in any of the generators t_{11}, t_{22} and t_{33}. Unfortunately, by re-ordering we may raise the degree in the diagonal generators; however, by analysing the commutation relations (2.3) we see that some of them (of the exchange type) are innocuous. Those which are of the commutator type and between the diagonal generators produce only the non-diagonal generators t_{ij} and lower the degree of *two* diagonal generators. Next, those of the commutator type between the non-diagonal generators produce at most one diagonal generator (namely t_{22}). Altogether, there is a net lowering in the complexity of the degree of the product $t_{11}t_{22}t_{33}$. This allows us to lower this degree consistently by iteration, until one of the diagonal generators is absent. Finally, by excluding the overlapping cases we arrive at the following result.

The independent basis in $sl_q(3)$ consist of three sectors composed by the elements $t^{\bar{n}}$ with one of the following restrictions:

$$\begin{aligned} &\text{(i)} \quad n_{11} = 0 \\ &\text{(ii)} \quad n_{11} \geqslant 1,\ n_{22} = 0 \\ &\text{(iii)} \quad n_{11} \geqslant 1,\ n_{22} \geqslant 1,\ n_{33} = 0. \end{aligned} \tag{3.4}$$

Now, by iterating the twisted derivation rules for $sl_q(3)$

$$\mathcal{L}(l_{ij}^{\pm})\varphi\psi = \sum_k \mathcal{L}(l_{kj}^{\pm})\varphi \cdot \mathcal{L}(l_{ik}^{\pm})\psi \tag{3.5}$$

$$\mathcal{R}(l_{ij}^{\pm})\varphi\psi = \sum_k \mathcal{R}(l_{ik}^{\pm})\varphi \cdot \mathcal{R}(l_{kj}^{\pm})\psi \tag{3.6}$$

we obtain the action of the representation $\mathcal{L}$ on $t^{\bar{n}}$ which we give in the appendix.

4. Reduction and intertwiners

We impose the conditions of the (infinitesimal) right covariance on the independent basis for $SL_q(3)$. First, we consider only those functions ψ (i.e. the multilabels $\bar{n}$) on which

$$\mathcal{R}(l_{ij}^{+})\psi = 0 \qquad i < j. \tag{4.1}$$

By these conditions some of the indices n_{ij} have to vanish, but for some others there is a possibility of compensation between various pieces, cf (A10)–(A12). This requires solutions to be in the form of an infinite series. Instead, we shall assume invertibility of some (special combinations of) generators t_{ij} and may recover the series by formal expansion of the inverses in q.

Note that (4.1) is not the only possible condition. There is a discrete choice of putting either X^+ or X^- equal to zero, corresponding to the lowest and highest weight modules respectively. In addition, there is a continuous choice of the Borel subgroup (or the set of positive roots); we expect that, for finite-dimensional groups, the results will not depend on this choice up to an equivalence.

We can also see that the solutions of (4.1) have to be built from generators t_{i3} in the last column and from minors m_{i1} of the first column (m_{ij} is the minor of t_{ij}, i.e. the q-determinant of T with ith line and jth column removed). Due to (2.4) one of the generators is dependent on the rest and we choose to delete m_{31}. Next, we impose that on ψ

$$\mathcal{R}(l_{ii}^+)\psi = \exp r_i \psi \tag{4.2}$$

for $i = 1, 2$ (the case $i = 3$ is not independent due to (2.12)), where the labelling of numbers r_1 and of r_2 will be specified further. We note the multiplicative property of the eigenvalue of $\mathcal{R}(l_{ii}^+)$ with respect to the product of eigenvectors and note that consequently the quotients $t_{i3}t_{j3}^{-1}$ and $m_{i1}m_{j1}^{-1}$ have eigenvalue one. Among them, we choose to work with the following three independent combinations:

$$w_1 = t_{13}t_{33}^{-1} \qquad w_2 = t_{23}t_{33}^{-1} \qquad w_3 = m_{21}m_{11}^{-1} \tag{4.3}$$

where $m_{21} = t_{12}t_{33} - qt_{13}t_{32}$ and $m_{11} = t_{22}t_{33} - qt_{23}t_{32}$.

Instead, in order to solve (4.2), we employ the combination $m_{11}^{j_1}t_{33}^{j_2}$, with j_1, j_2 to be determined in terms of r_1, r_2. Indeed, we obtain

$$r_1 = -(j_2 + 2j_1)h/3 \qquad r_2 = -(j_1 + j_2)h/3$$

where $q = \exp h$. Thus, since $j_1, j_2 \in \mathbb{Z}$, the admissible values of r_1 and of r_2 have to be quantized in units of $h/3$. Finally, the basis in the space $\mathcal{T}_{\bar{j}}$, $\bar{j} =: (j_1, j_2)$, of the common solutions of (4.1) and (4.2) consist of the ordered monomials

$$\phi_{\bar{j}}^{\bar{n}} =: w_1^{n_1} w_2^{n_2} w_3^{n_3} m_{11}^{j_1} t_{33}^{j_2} \tag{4.4}$$

where $n_1, n_2, n_3 \in \mathbb{Z}_+$. In this way we obtain an infinite family of reduced representation spaces $\mathcal{T}_{\bar{j}}$, indexed by the integers j_1, j_2.

It is remarkable that the three independent variables w_i form a closed algebra:

$$w_1w_2 = qw_2w_1 \qquad w_1w_3 = q^{-1}w_3w_1 \qquad w_2w_3 = qw_3w_2 + \lambda w_1. \tag{4.5}$$

We also give some relations useful for the computations that follow:

$$w_3^n w_2 = q^{-n}w_2w_3^n + \lambda q^{-1}[n]w_1w_3^{n-1} \qquad w_3w_2^n = q^{-n}w_2^n w_3 + \lambda q^{-n}[n]w_1w_2^{n-1}. \tag{4.6}$$

Moreover, the two 'spectators' t_{33} and m_{11} commute:

$$t_{33}m_{11} = m_{11}t_{33} \tag{4.7}$$

and have the following commutation relations with the variables w_i:

$$w_1m_{11} = qm_{11}w_1 \qquad w_2m_{11} = m_{11}w_2 \qquad w_3m_{11} = qm_{11}w_3 \tag{4.8}$$

$$w_1t_{33} = qt_{33}w_1 \qquad w_2t_{33} = qt_{33}w_2 \qquad w_3t_{33} = t_{33}w_3. \tag{4.9}$$

We note that our w_i are also the relevant variables as far as the global covariance is concerned. This can be seen from the Gauss decomposition of T:

$$\begin{pmatrix}1 & w_3 & w_1\\ 0 & 1 & w_2\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}t_{11}-w_3t_{33}^{-1}m_{12}-w_1t_{31} & 0 & 0\\ 0 & t_{33}^{-1}m_{11} & 0\\ 0 & 0 & t_{33}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ m_{11}^{-1}m_{12} & 1 & 0\\ t_{33}^{-1}t_{31} & t_{33}^{-1}t_{32} & 1\end{pmatrix}. \tag{4.10}$$

By a lengthy but straightforward computation, iterating the twisted derivation rules, we obtain the explicit formulae for the representation in $\mathcal{T}_{\bar{j}}$:

$$\mathcal{L}(l_{12}^+)\phi_{\bar{j}}^{\bar{n}} = -\lambda q^{1-n_2-n_3+(2j_1+j_2)/3}[n_2]\phi_{\bar{j}}^{n_1,n_2-1,n_3} + \lambda q^{1-n_2+(j_2-j_1)/3}[n_3-j_1]\phi_{\bar{j}}^{n_1,n_2,n_3+1} \tag{4.11}$$

$$\mathcal{L}(l_{23}^+)\phi_{\bar{j}}^{\bar{n}} = \lambda q^{1-(2j_2+j_1)/3}[n_1+n_2-n_3-j_2]\phi_{\bar{j}}^{n_1,n_2+1,n_3} + \lambda q^{n_1-(2j_2+j_1)/3}[n_3]\phi_{\bar{j}}^{n_1+1,n_2,n_3-1} \tag{4.12}$$

$$\mathcal{L}(l_{13}^+)\phi_{\bar{j}}^{\bar{n}} = \lambda q^{1-n_2+(4j_2-j_1)/3}[n_1-n_2-n_3-j_1+j_2]\phi_{\bar{j}}^{n_1+1,n_2,n_3} + \lambda q^{2-n_1-n_2+(j_2-j_1)/3}[j_1-n_3]\phi_{\bar{j}}^{n_1,n_2+1,n_3+1} \tag{4.13}$$

$$\mathcal{L}(l_{11}^+)\phi_{\bar{j}}^{\bar{n}} = q^{-n_1-n_3+(2j_1+j_2)/3}\phi_{\bar{j}}^{\bar{n}} \tag{4.14}$$

$$\mathcal{L}(l_{22}^+)\phi_{\bar{j}}^{\bar{n}} = q^{-n_2+n_3+(j_2-j_1)/3}\phi_{\bar{j}}^{\bar{n}} \tag{4.15}$$

$$\mathcal{L}(l_{33}^+)\phi_{\bar{j}}^{\bar{n}} = q^{n_1+n_2-(j_1+2j_2)/3}\phi_{\bar{j}}^{\bar{n}} \tag{4.16}$$

$$\mathcal{L}(l_{21}^-)\phi_{\bar{j}}^{\bar{n}} = \lambda q^{n_1-1+(j_1-j_2)/3}[n_3]\phi_{\bar{j}}^{n_1,n_2,n_3-1} + \lambda q^{n_2-n_3+(j_1-j_2)/3}[n_1]\phi_{\bar{j}}^{n_1-1,n_2+1,n_3} \tag{4.17}$$

$$\mathcal{L}(l_{32}^-)\phi_{\bar{j}}^{\bar{n}} = \lambda q^{-1+(j_1+2j_2)/3}[n_2]\phi_{\bar{j}}^{n_1,n_2-1,n_3} \tag{4.18}$$

$$\mathcal{L}(l_{31}^-)\phi_{\bar{j}}^{\bar{n}} = \lambda q^{n_2-1+(j_1+2j_2)/3}[n_1]\phi_{\bar{j}}^{n_1-1,n_2,n_3}. \tag{4.19}$$

So far we have restricted the left regular representation to (infinite dimensional) subrepresentations $\mathcal{T}_{\bar{j}}$. The restricted functions depend, effectively, only on the variables w_i since for each fixed $\bar{j}$ the factor $m_{11}^{j_1}t_{33}^{j_2}$ is fixed. In order to pursue the reduction to the end and to obtain the functions depending *manifestly* on w_i only, we associate with each $\phi \in T_j$ a function $\hat{\phi}$, by the formula

$$\hat{\phi} = \phi m_{11}^{-j_1}t_{33}^{-j_2}. \tag{4.20}$$

We note that we actually have a freedom to work with variables proportional to w_i:

$$z_i = C_i w_i \tag{4.21}$$

where C_i are constants depending on q and $\bar{j}$. (This leads to an equivalent representation.) In order to simplify the formulae we choose

$$C_1 = C_2 = q^{1-(j_1+2j_2)/3} \qquad C_3 = q^{1+(j_2-j_1)/3}. \tag{4.22}$$

The representations (4.11)–(4.19) acting in $\mathcal{T}_{\bar{j}}$ induce the following transformation rules on the basis $z_1^{n_1}z_2^{n_2}z_3^{n_3}$:

$$\hat{\mathcal{L}}(l_{12}^+)z_1^{n_1}z_2^{n_2}z_3^{n_3} = -\lambda q^{2-n_2-n_3+(j_1-j_2)/3}[n_2]z_1^{n_1}z_2^{n_2-1}z_3^{n_3} + \lambda q^{-n_2}[n_3-j_1]z_1^{n_1}z_2^{n_2}z_3^{n_3+1} \tag{4.23}$$

$$\hat{\mathcal{L}}(l_{23}^+)z_1^{n_1}z_2^{n_2}z_3^{n_3} = \lambda q^{n_1+(j_2-j_1)/3}[n_3]z_1^{n_1+1}z_2^{n_2}z_3^{n_3-1} + \lambda[n_1+n_2-n_3-j_2]z_1^{n_1}z_2^{n_2+1}z_3^{n_3} \tag{4.24}$$

$$\hat{\mathcal{L}}(l_{13}^+)z_1^{n_1}z_2^{n_2}z_3^{n_3} = \lambda q^{-n_1-n_2+(2j_2+j_1)/3}[j_1-n_3]z_1^{n_1}z_2^{n_2+1}z_3^{n_3+1} + \lambda q^{-n_2+2j_2}[n_1-n_2-n_3-j_1+j_2]z_1^{n_1+1}z_2^{n_2}z_3^{n_3} \tag{4.25}$$

$$\hat{\mathcal{L}}(l_{11}^+)z_1^{n_1}z_2^{n_2}z_3^{n_3} = q^{-n_1-n_3+(2j_1+j_2)/3}z_1^{n_1}z_2^{n_2}z_3^{n_3} \tag{4.26}$$

$$\hat{\mathcal{L}}(l_{22}^+)z_1^{n_1}z_2^{n_2}z_3^{n_3} = q^{-n_2+n_3+(j_2-j_1)/3}z_1^{n_1}z_2^{n_2}z_3^{n_3} \tag{4.27}$$

$$\hat{\mathcal{L}}(l_{33}^+)z_1^{n_1}z_2^{n_2}z_3^{n_3} = q^{n_1+n_2-(j_1+2j_2)/3}z_1^{n_1}z_2^{n_2}z_3^{n_3} \tag{4.28}$$

$$\hat{\mathcal{L}}(l_{21}^-)z_1^{n_1}z_2^{n_2}z_3^{n_3} = \lambda q^{n_1}[n_3]z_1^{n_1}z_2^{n_2}z_3^{n_3-1} + \lambda q^{n_2+n_3+(j_1-j_2)/3}[n_1]z_1^{n_1-1}z_2^{n_2}z_3^{n_3} \tag{4.29}$$

$$\hat{\mathcal{L}}(l_{32}^-)z_1^{n_1}z_2^{n_2}z_3^{n_3} = \lambda[n_2]z_1^{n_1}z_2^{n_2-1}z_3^{n_3} \tag{4.30}$$

$$\hat{\mathcal{L}}(l_{31}^-)z_1^{n_1}z_2^{n_2}z_3^{n_3} = \lambda q^{n_2}[n_1]z_1^{n_1-1}z_2^{n_2}z_3^{n_3}. \tag{4.31}$$

These expressions define the representations $\hat{\mathcal{L}}_{\bar{j}}$ which act in the (same for all $\bar{j}$) space $\hat{\mathcal{T}}_{\bar{j}}$ of polynomials $\hat{\phi}$ in z_i (which as a linear space is just T_{00}). (Note, however, that the concrete form of z_i depends on $\bar{j}$; we do not indicate this dependence explicitly in order to simplify the notation.) Using the operations M_i, rescaling the variable $z_i \to qz_i$, the q-derivative $D_i f = (M_i - M_i^{-1})f/(q-q^{-1})z$ and the operator Z_i and multiplying by the variable z_i, all of which by convention act *directly* on the ith place in the ordered monomials, we obtain the following reduced q-differential representation:

$$\hat{\mathcal{L}}(l_{12}^+) = -\lambda q^{2+(j_1-j_2)/3}M_2^{-1}M_3^{-1}D_2 + \lambda(q^{-j_1}M_3 - q^{j_1}M_3^{-1})M_2^{-1}Z_3 \tag{4.32}$$

$$\hat{\mathcal{L}}(l_{23}^+) = -\lambda q^{(j_2-j_1)/3}M_1Z_1D_3 + \lambda(q^{-j_2}M_1^{-1}M_2^{-1}M_3 - q^{j_2}M_1M_2M_3^{-1})Z_2 \tag{4.33}$$

$$\hat{\mathcal{L}}(l_{13}^+) = \lambda q^{(2j_2+j_1)/3}(q^{j_1}M_3^{-1} - q^{j_1}M_3)M_1^{-1}M_2^{-1}Z_2Z_3 + \lambda q^{2j_2}(q^{j_2-j_1}M_1M_2^{-1}M_3^{-1} - q^{j_1-j_2}M_1^{-1}M_2M_3)M_2^{-1}Z_1 \tag{4.34}$$

$$\hat{\mathcal{L}}(l_{11}^+) = q^{(2j_1+j_2)/3}M_1^{-1}M_3^{-1} \tag{4.35}$$

$$\hat{\mathcal{L}}(l_{22}^+) = q^{(j_2-j_1)/3}M_2^{-1}M_3 \tag{4.36}$$

$$\hat{\mathcal{L}}(l_{33}^+) = q^{-(j_1+2j_2)/3}M_1M_2 \tag{4.37}$$

$$\hat{\mathcal{L}}(l_{21}^-) = \lambda M_1D_3 + \lambda q^{(j_1-j_2)/3}M_2M_3^{-1}D_1 \tag{4.38}$$

$$\hat{\mathcal{L}}(l_{32}^-) = \lambda D_2 \tag{4.39}$$

$$\hat{\mathcal{L}}(l_{31}^-) = \lambda M_2D_1. \tag{4.40}$$

We remark that although originally we assumed $j_1, j_2 \in \mathbb{Z}$, the formulae (4.32)–(4.40) also define a representation $\hat{\mathcal{T}}_{\bar{j}}$ of $sl_q(3)$ for arbitrary $j_1, j_2 \in \mathbb{C}$; however, only for $j_1, j_2 \in \mathbb{Z}$ do they comprise of a deformation of representations which are integrable to a representation of the group $SL(3, \mathbb{C})$.

Now we consider the right regular representation (A16)–(A18) of the remaining generators on $\phi^{\bar{n}}_{\bar{j}}$ and restrict it. First, we verify when these representations are defined as operators between some of the spaces $\mathcal{T}_{\bar{j}}$. It turns out that for any j_1, j_2 the image of $\mathcal{R}(l^-_{31})$ always contains non-admissible terms like m_{12} or m_{22} and we do not consider this operator any further. Instead, we have

$$\mathcal{R}(l^-_{21})\phi^{\bar{n}}_{\bar{j}} = \lambda q^{(2j_1-j_1)/3}(q^{n_3-1}[n_3]w_1^{n_1}w_2^{n_2}w_3^{n_3-1}m_{11}^{j_1-2}t_{33}^{j_2+1} - [j_1]w_1^{n_1}w_2^{n_2}w_3^{n_3}m_{11}^{j_1-1}t_{33}^{j_2}m_{12}) \tag{4.41}$$

$$\mathcal{R}(l^-_{32})\phi^{\bar{n}}_{\bar{j}} = -\lambda q^{(j_2-j_1)/3}(q^{n_1+n_2-n_3-1}[n_1]w_1^{n_1-1}w_2^{n_2}w_3^{n_3+1}m_{11}^{j_1+1}t_{33}^{j_2-2} + q^{2n_1+n_2-n_3-1}[n_2]w_1^{n_1}w_2^{n_2-1}w_3^{n_3}m_{11}^{j_1+1}t_{33}^{j_2-2} + [j_2]w_1^{n_1}w_2^{n_2}w_3^{n_3}m_{11}^{j_1}t_{33}^{j_2-1}t_{32}). \tag{4.42}$$

Thus for $\mathcal{R}(l^-_{21})$ and for $\mathcal{R}(l^-_{32})$ the non-admissible terms (with t_{32} or m_{12}) are not present precisely when $j_1 = 0$ and $j_2 = 0$, respectively. Then, in fact, we obtain two intertwiners: the restriction of $\mathcal{R}(l^-_{21})$ to $\mathcal{T}_{0,j_2}$ which maps into $\mathcal{T}_{-2,j_2+1}$ and the restriction of $\mathcal{R}(l^-_{32})$ to $\mathcal{T}_{j_1,0}$ which maps into $\mathcal{T}_{j_1+1,-2}$. By similar computations (first for the squares of these operators and then by induction) one obtains a family of other intertwiners

$$\mathcal{R}(l^-_{21})^{j_1+1} : \mathcal{T}_{j_1,j_2} \to \mathcal{T}_{-j_1-2,j_2+j_1+1} \qquad \text{for } j_1 \in \mathbb{Z}_+, j_2 \in \mathbb{C} \tag{4.43}$$

$$\mathcal{R}(l^-_{32})^{j_2+1} : \mathcal{T}_{j_1,j_2} \to \mathcal{T}_{j_1+j_2+1,-j_2-2} \qquad \text{for } j_2 \in \mathbb{Z}_+, j_1 \in \mathbb{C}. \tag{4.44}$$

There are also some mixed intertwiners:

$$\mathcal{R}(l^-_{32})^{j_2+1}\mathcal{R}(l^-_{21})^{j_1+1} : \mathcal{T}_{j_1,j_2-j_1-1} \to \mathcal{T}_{j_2-j_1-1,-j_2-2} \qquad \text{for } j_1, j_2 \in \mathbb{Z}_+ \tag{4.45}$$

$$\mathcal{R}(l^-_{21})^{j_1+1}\mathcal{R}(l^-_{32})^{j_2+1} : \mathcal{T}_{j_1-j_2-1,j_2} \to \mathcal{T}_{-j_1-2,j_1-j_2-1} \qquad \text{for} j_1, j_2 \in \mathbb{Z}_+. \tag{4.46}$$

In the same manner as that used for the left-represented operators we perform the explicit reduction to the space $\hat{\mathcal{T}}_{\bar{j}}$ of functions of the variables z_i. Keeping track of the changes of indices $\bar{n}$ and of the labels $\bar{j}$ on which the coefficients C_i depend, we obtain the two basic intertwiners

$$\hat{\mathcal{R}}(l^-_{21})z_1^{n_1}z_2^{n_2}z_3^{n_3} = \lambda q^{1+2j_2/3}D_3 \tag{4.47}$$

$$\hat{\mathcal{R}}(l^-_{32})z_1^{n_1}z_2^{n_2}z_3^{n_3} = -\lambda q^{1-j_1/3}D_1Z_3 - \lambda q^{1-2j_1/3}M_1D_2. \tag{4.48}$$

The other intertwiners can easily be obtained by observing that the explicit formula for the intertwiner corresponding to the product of (any number of) l^-_{21} and l^-_{32} is just the product of the basic intertwiners

$$\hat{\mathcal{R}}\left((l^-_{32})^{j_2}(l^-_{21})^{j_1}\right) = \left(\hat{\mathcal{R}}(l^-_{32})\right)^{j_2}\left(\hat{\mathcal{R}}(l^-_{21})\right)^{j_1}. \tag{4.49}$$

They clearly satisfy

$$\left(\hat{\mathcal{R}}(l^-_{21})\right)^{j_1+1} \cdot \hat{\mathcal{L}}_{j_1,j_2}(l^\pm_{k\ell}) = \hat{\mathcal{L}}_{-j_1-2,j_2+j_1+1}(l^\pm_{k\ell}) \cdot \left(\hat{\mathcal{R}}(l^-_{21})\right)^{j_1+1} \tag{4.50}$$

$$\left(\hat{\mathcal{R}}(l^-_{32})\right)^{j_2+1} \cdot \hat{\mathcal{L}}_{j_1,j_2}(l^\pm_{k\ell}) = \hat{\mathcal{L}}_{j_1+j_2+1,-j_2-2}(l^\pm_{k\ell}) \cdot \left(\hat{\mathcal{R}}(l^-_{32})\right)^{j_2+1} \tag{4.51}$$

$$\hat{\mathcal{R}}\left((l_{21}^{-})^{j_1+1}(l_{32}^{-})^{j_2+1}\right)\cdot\hat{\mathcal{L}}_{j_1-j_2-1,j_2}(l_{k\ell}^{\pm})=\hat{\mathcal{L}}_{-j_1-2,j_1-j_2-1}(l_{k\ell}^{\pm})\cdot\hat{\mathcal{R}}\left((l_{21}^{-})^{j_1+1}(l_{32}^{-})^{j_2+1}\right) \quad (4.52)$$

$$\hat{\mathcal{R}}\left((l_{32}^{-})^{j_2+1}(l_{21}^{-})^{j_1+1}\right)\cdot\hat{\mathcal{L}}_{j_1,j_2-j_1-1}(l_{k\ell}^{\pm})=\hat{\mathcal{L}}_{j_2-j_1-1,-j_2-2}(l_{k\ell}^{\pm})\cdot\hat{\mathcal{R}}\left((l_{32}^{-})^{j_2+1}(l_{21}^{-})^{j_1+1}\right) \quad (4.53)$$

where we have explicitly indicated the labels of the representations $\hat{\mathcal{L}}$. These intertwiners give rise to various partial equivalences between some $\hat{\mathcal{T}}_{\bar{j}}$. In particular, the kernels of these intertwiners form invariant subrepresentations. Thus, for generic $j_1, j_2 \in \mathbb{C}$ the representations $\hat{\mathcal{T}}_{\bar{j}}$ are irreducible except for $j_1 \in \mathbb{Z}_+$ or $j_2 \in \mathbb{Z}_+$, when they are reducible. To further study the reducibility, we note that all the intertwiners which act on $\hat{\mathcal{T}}_{\bar{j}}$ for given $j_1, j_2 \in \mathbb{Z}_+$, contain as a last factor either the j_1+1 power of the basic intertwiner (4.47) or the j_2+1 power of (4.48). Thus, it is sufficient to consider only the kernels of the two intertwiners (4.50) and (4.51). It turns out that their common intersections are irreducible (sub)representations and have the dimension $d = (j_1+1)(j_2+1)\,(1+(j_1+j_2)/2)$. They consist of polynomials in z_i of limited order, depending on i and on $\bar{j}$, and are q-deformations of the well known representations of $sl(3)$. In particular, $\{1, z_1, z_2\}$ is a basis of $3_q = \ker\hat{\mathcal{R}}(l_{21}^{-}) \cap \ker\left(\hat{\mathcal{R}}(l_{32}^{-})\right)^2 \subset \hat{\mathcal{T}}_{0,1}$, $\{1, z_3, z_2z_3 - q^{-1/3}z_1\}$ is a basis of $3_q^* = \ker\left(\hat{\mathcal{R}}(l_{21}^{-})\right)^2 \cap \ker\hat{\mathcal{R}}(l_{32}^{-}) \subset \hat{\mathcal{T}}_{1,0}$ and $\{1, z_1, z_2, z_3, z_1z_3, z_2z_3, q^{1/3}(1+q)z_2^2z_3 - [2]z_1z_2, q^{1/3}[2]z_1z_2z_3 - (1+q)z_1^2\}$ is a basis of $8_q = \ker\left(\hat{\mathcal{R}}(l_{21}^{-})\right)^2 \cap \ker\left(\hat{\mathcal{R}}(l_{32}^{-})\right)^2 \subset \hat{\mathcal{T}}_{1,1}$.

5. Final remarks

In this paper, the method previously applied to simpler examples has been extended to $sl_q(3)$. This permits us to obtain, in a canonical way, some important (reducible and irreducible) representations of $sl_q(3)$ and their intertwiners which present some new features with respect to the $sl_q(2)$; namely, in $sl_q(2)$ the commuting algebra of the intertwiners has just one generator, while in $sl_q(3)$ we obtain three non-commuting difference operators. Moreover, $sl_q(3)$ is the first case in which the space of representations becomes structured, that is, the Weyl chamber has edges and interior parts corresponding to the different homogeneous spaces; consequently not all representations are generated as tensor powers of a single representation.

The reduction procedure corresponds, in fact, to working on the quantum three-dimensional complex flag manifold $F_q(1,2;3)$ (the quotient manifold of $SL_q(3)$ by the Borel subgroup), cf [12, 13]. This is indeed the case for the representations inside the Weyl chamber ($j_1 \cdot j_2 \neq 0$), while the representations on the edges (j_1 or $j_2 = 0$) live, as in the previous simpler examples, over the q-projective spaces (or q-Grassmannians). Our functions are local representatives of global sections, like e.g. $m_{11}^{j_1}t_{33}^{j_2}$, of some quantum line bundles over $F_q(1,2;3)$. This holds over one particular patch, coordinated by the variables z_i (or w_i), corresponding to the requirement of invertibility of m_{11} and t_{33}.

Our work relates to the Borel–Weil theory which links the irreducible representations of groups to the spaces of sections of certain bundles over related quotient spaces. A q-deformation of this theory [14] is not fully satisfactory from the geometric point of view as it mainly uses purely analytic and algebraic methods. Our approach should be useful for that purpose and provides concrete examples which can be used as a test for choosing a proper definition of what an abstract quantum bundle should be (which still lacks full agreement).

We have also followed the original idea with the modification that we have used the infinitesimal covariance under g_q rather than the global one to induce the character of the Borel subgroup. Though closely related, this seems to be conceptually simpler than identifying what the q-analogue of induced representations should be (which would require global covariance under the co-representations of G_q), see for example [15–18]. A detailed comparison of these two methods and the geometry of the quantum homogeneous space $F_q(1, 2; 3)$ will be discussed in a forthcoming paper.

We intend to extend the above programme to the general case of $sl_q(n)$; this requires a more convenient working form of the exchange relations for the flag coordinates, permitting us to order them in a consistent way. A relevant work [19], about which we were informed after submitting this paper, completes the above programme for $U_q(sl(3))$ only and has an overlap with our independent results.

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Appendix

Denote $s = \frac{1}{3}(n_{11}+n_{12}+\ldots+n_{33})$. The action of the representation $\mathcal{L}$ on $t^{\bar{n}} = t_{11}^{n_{11}} t_{12}^{n_{12}} \ldots t_{33}^{n_{33}}$ is:

$$\begin{aligned}\mathcal{L}(l_{12}^{+})t^{\bar{n}} = -\lambda q^{s} \Big(& q^{1-n_{12}-n_{13}-n_{21}} [n_{21}] t_{11}^{n_{11}+1} \ldots t_{21}^{n_{21}-1} \ldots t_{33}^{n_{33}} \\ & + q^{1-n_{21}-n_{22}-n_{13}} [n_{22}] t_{11}^{n_{11}} t_{12}^{n_{12}+1} \ldots t_{22}^{n_{22}-1} \ldots t_{33}^{n_{33}} \\ & + q^{1-n_{21}-n_{22}-n_{23}} [n_{23}] t_{11}^{n_{11}} \ldots t_{13}^{n_{13}+1} \ldots t_{23}^{n_{23}-1} \ldots t_{33}^{n_{33}} \Big)\end{aligned} \tag{A1}$$

$$\begin{aligned}\mathcal{L}(l_{23}^{+})t^{\bar{n}} = -\lambda q^{s} \Big(& q^{1-n_{22}-n_{23}-n_{31}} [n_{21}] t_{11}^{n_{11}} \ldots t_{21}^{n_{21}+1} \ldots t_{31}^{n_{31}-1} \ldots t_{33}^{n_{33}} \\ & + q^{1-n_{23}-n_{31}-n_{32}} [n_{32}] t_{11}^{n_{11}} \ldots t_{22}^{n_{22}+1} \ldots t_{32}^{n_{32}-1} t_{33}^{n_{33}} \\ & + q^{1-n_{21}-n_{32}-n_{33}} [n_{33}] t_{11}^{n_{11}} \ldots t_{23}^{n_{23}+1} \ldots t_{33}^{n_{33}-1} \Big)\end{aligned} \tag{A2}$$

$$\begin{aligned}\mathcal{L}(l_{13}^{+})t^{\bar{n}} = -\lambda q^{1-n_{31}+s} \Big(& q^{-n_{12}-n_{13}-n_{21}} [n_{31}] t_{11}^{n_{11}+1} \ldots t_{31}^{n_{31}-1} \ldots t_{33}^{n_{33}} \\ & + q^{-n_{13}-n_{22}-n_{32}} [n_{32}] t_{11}^{n_{11}} t_{12}^{n_{12}+1} \ldots t_{32}^{n_{32}-1} t_{33}^{n_{33}} \\ & + q^{-n_{23}-n_{32}-n_{33}} [n_{33}] t_{11}^{n_{11}} \ldots t_{13}^{n_{13}+1} \ldots t_{33}^{n_{33}-1} \Big) \\ + t^{2} q^{1-n_{31}+s} \Big(& q^{1-n_{21}-n_{22}-n_{13}-n_{12}} [n_{31}][n_{22}] t_{11}^{n_{11}+1} \ldots t_{21}^{n_{21}+1} t_{22}^{n_{22}-1} \ldots t_{31}^{n_{31}-1} \ldots t_{33}^{n_{33}} \\ & + q^{1-n_{22}-n_{23}} [n_{31}][n_{23}] t_{11}^{n_{11}} \ldots t_{13}^{n_{13}+1} t_{21}^{n_{21}+1} \ldots t_{23}^{n_{23}-1} \ldots t_{31}^{n_{31}-1} \ldots t_{33}^{n_{33}} \\ & + q^{1-n_{23}-n_{32}} [n_{23}][n_{32}] t_{11}^{n_{11}} \ldots t_{13}^{n_{13}+1} \ldots t_{22}^{n_{22}+1} t_{23}^{n_{23}-1} \ldots t_{32}^{n_{32}-1} t_{33}^{n_{33}} \Big)\end{aligned} \tag{A3}$$

$$\mathcal{L}(l_{11}^{+})t^{\bar{n}} = q^{s-n_{11}-n_{12}-n_{13}} t^{\bar{n}} \tag{A4}$$

$$\mathcal{L}(l_{22}^{+})t^{\bar{n}} = q^{s-n_{21}-n_{22}-n_{23}} t^{\bar{n}} \tag{A5}$$

$$\mathcal{L}(l_{33}^{+})t^{\bar{n}} = q^{s-n_{31}-n_{32}-n_{33}}t^{\bar{n}} \tag{A6}$$

$$\mathcal{L}(l_{21}^{-})t^{\bar{n}} = \lambda q^{-s}\Big(q^{n_{11}+n_{12}+n_{23}}[n_{13}]t_{11}^{n_{11}}\dots t_{13}^{n_{13}-1}\dots t_{23}^{n_{23}+1}t_{33}^{n_{33}} + q^{n_{11}+n_{22}+n_{23}}[n_{12}]t_{11}^{n_{11}}t_{12}^{n_{12}-1}\dots t_{22}^{n_{22}+1}\dots t_{33}^{n_{33}} + q^{n_{21}+n_{22}+n_{23}}[n_{11}]t_{11}^{n_{11}-1}\dots t_{21}^{n_{21}+1}\dots t_{33}^{n_{33}}\Big) \tag{A7}$$

$$\mathcal{L}(l_{32}^{-})t^{\bar{n}} = \lambda q^{-s}\Big(q^{n_{21}+n_{22}+n_{33}}[n_{23}]t_{11}^{n_{11}}\dots t_{23}^{n_{23}-1}\dots t_{33}^{n_{33}+1} + q^{n_{21}+n_{32}+n_{33}}[n_{22}]t_{11}^{n_{11}}\dots t_{22}^{n_{22}-1}\dots t_{32}^{n_{32}+1}t_{33}^{n_{33}} + q^{n_{31}+n_{32}+n_{33}}[n_{21}]t_{11}^{n_{11}}\dots t_{21}^{n_{21}-1}\dots t_{31}^{n_{31}+1}\dots t_{33}^{n_{33}}\Big) \tag{A8}$$

$$\begin{aligned}\mathcal{L}(l_{31}^{-})t^{\bar{n}} = {}& \lambda q^{n_{33}-s}\Big(q^{n_{11}+n_{12}+n_{23}}[n_{13}]t_{11}^{n_{11}}\dots t_{13}^{n_{13}-1}\dots t_{33}^{n_{33}+1} \\ &+ q^{n_{11}+n_{22}+n_{32}}[n_{12}]t_{11}^{n_{11}}t_{12}^{n_{12}-1}\dots t_{32}^{n_{32}+1}t_{33}^{n_{33}} \\ &+ q^{n_{21}+n_{31}-n_{32}}[n_{11}]t_{11}^{n_{11}-1}\dots t_{31}^{n_{31}+1}\dots t_{33}^{n_{33}}\Big) \\ &+ \lambda^2 q^{n_{33}-s}\Big(q^{n_{11}+n_{22}}[n_{12}][n_{23}]t_{11}^{n_{11}}\dots t_{12}^{n_{12}-1}\dots t_{22}^{n_{22}+1}t_{23}^{n_{23}-1}\dots t_{33}^{n_{33}+1} \\ &+ q^{n_{21}+n_{22}}[n_{11}][n_{23}]t_{11}^{n_{11}-1}\dots t_{21}^{n_{21}+1}\dots t_{23}^{n_{23}-1}\dots t_{33}^{n_{33}+1} \\ &+ q^{n_{21}+n_{32}}[n_{11}][n_{22}]t_{11}^{n_{11}-1}\dots t_{21}^{n_{23}+1}t_{22}^{n_{22}-1}\dots t_{32}^{n_{32}+1}t_{33}^{n_{33}}\Big)\end{aligned} \tag{A9}$$

$$\mathcal{R}(l_{12}^{+})t^{\bar{n}} = \lambda q^{n_{32}-s}\Big(q^{n_{11}+n_{21}}[n_{31}]t_{11}^{n_{11}}\dots t_{31}^{n_{31}-1}t_{32}^{n_{32}+1}t_{33}^{n_{33}} + q^{n_{11}+n_{22}}[n_{21}]t_{11}^{n_{11}}\dots t_{21}^{n_{21}-1}t_{22}^{n_{22}+1}\dots t_{33}^{n_{33}} + q^{n_{12}+n_{22}}[n_{11}]t_{11}^{n_{11}-1}t_{12}^{n_{12}+1}\dots t_{33}^{n_{33}}\Big) \tag{A10}$$

$$\mathcal{R}(l_{23}^{+})t^{\bar{n}} = \lambda q^{n_{33}-s}\Big(q^{n_{12}+n_{22}}[n_{32}]t_{11}^{n_{11}}\dots t_{32}^{n_{32}-1}t_{33}^{n_{33}+1} + q^{n_{12}+n_{23}}[n_{22}]t_{11}^{n_{11}}\dots t_{22}^{n_{22}-1}t_{23}^{n_{23}+1}\dots t_{33}^{n_{33}} + q^{n_{13}+n_{23}}[n_{12}]t_{11}^{n_{11}}t_{12}^{n_{12}-1}t_{13}^{n_{13}+1}\dots t_{33}^{n_{33}}\Big) \tag{A11}$$

$$\begin{aligned}\mathcal{R}(l_{13}^{+})t^{\bar{n}} = {}& \lambda q^{n_{33}-s}\Big(q^{n_{11}+n_{21}+n_{32}}[n_{31}]t_{11}^{n_{11}}\dots t_{31}^{n_{31}-1}\dots t_{33}^{n_{33}+1} \\ &+ q^{n_{11}+n_{22}+n_{23}}[n_{21}]t_{11}^{n_{11}}\dots t_{21}^{n_{21}-1}\dots t_{23}^{n_{23}+1}\dots t_{33}^{n_{33}} \\ &+ q^{n_{13}+n_{12}+n_{23}}[n_{11}]t_{11}^{n_{11}-1}\dots t_{13}^{n_{13}+1}\dots t_{33}^{n_{33}}\Big) \\ &+ \lambda^2 q^{n_{33}-s}\Big(q^{n_{11}+n_{22}}[n_{21}][n_{32}]t_{11}^{n_{11}}\dots t_{21}^{n_{21}-1}t_{22}^{n_{22}+1}\dots t_{32}^{n_{32}-1}\dots t_{33}^{n_{33}+1} \\ &+ q^{n_{12}+n_{22}}[n_{11}][n_{32}]t_{11}^{n_{11}-1}t_{12}^{n_{12}+1}\dots t_{32}^{n_{32}-1}t_{33}^{n_{33}+1} \\ &+ q^{n_{12}+n_{23}}[n_{11}][n_{22}]t_{11}^{n_{11}-1}t_{12}^{n_{12}+1}\dots t_{22}^{n_{22}-1}t_{23}^{n_{23}+1}\dots t_{33}^{n_{33}}\Big)\end{aligned} \tag{A12}$$

$$\mathcal{R}(l_{11}^{+})t^{\bar{n}} = q^{-s+n_{11}+n_{21}+n_{31}}t^{\bar{n}} \tag{A13}$$

$$\mathcal{R}(l_{22}^{+})t^{\bar{n}} = q^{s+n_{12}+n_{22}+n_{33}}t^{\bar{n}} \tag{A14}$$

$$\mathcal{R}(l_{33}^{+})t^{\bar{n}} = q^{s+n_{13}+n_{23}+n_{33}}t^{\bar{n}} \tag{A15}$$

$$\begin{aligned}\mathcal{R}(l_{21}^{-})t^{\bar{n}} = -\lambda q^{1-n_{12}+s}\Big(&q^{-n_{21}-n_{31}}[n_{12}]t_{11}^{n_{11}+1}t_{12}^{n_{12}-1}\dots t_{33}^{n_{33}}\\&+q^{-n_{22}-n_{31}}[n_{22}]t_{11}^{n_{11}}\dots t_{21}^{n_{21}+1}t_{22}^{n_{22}-1}\dots t_{33}^{n_{33}}\\&+q^{-n_{22}-n_{32}}[n_{32}]t_{11}^{n_{11}}\dots t_{31}^{n_{31}+1}\dots t_{32}^{n_{32}-1}t_{33}^{n_{33}}\Big)\end{aligned} \tag{A16}$$

$$\begin{aligned}\mathcal{R}(l_{32}^{-})t^{\bar{n}} = -\lambda q^{1-n_{13}+s}\Big(&q^{-n_{22}-n_{32}}[n_{13}]t_{11}^{n_{11}}t_{12}^{n_{12}+1}t_{13}^{n_{13}-1}\dots t_{33}^{n_{33}}\\&+q^{-n_{23}-n_{32}}[n_{23}]t_{11}^{n_{11}}\dots t_{22}^{n_{22}+1}t_{23}^{n_{23}-1}\dots t_{33}^{n_{33}}\\&+q^{-n_{23}-n_{33}}[n_{33}]t_{11}^{n_{11}}\dots t_{32}^{n_{32}+1}t_{33}^{n_{33}-1}\Big)\end{aligned} \tag{A17}$$

$$\begin{aligned}\mathcal{R}(l_{31}^{-})t^{\bar{n}} = -\lambda q^{1-n_{13}+s}\Big(&q^{-n_{12}-n_{21}-n_{31}}[n_{13}]t_{11}^{n_{11}+1}\dots t_{13}^{n_{13}-1}\dots t_{33}^{n_{33}}\\&+q^{-n_{22}-n_{23}-n_{31}}[n_{23}]t_{11}^{n_{11}}\dots t_{21}^{n_{21}+1}\dots t_{23}^{n_{23}-1}\dots t_{33}^{n_{33}}\\&+q^{-n_{23}-n_{32}-n_{33}}[n_{33}]t_{11}^{n_{11}}\dots t_{31}^{n_{31}+1}\dots t_{33}^{n_{33}-1}\Big)\\&+\lambda^{2}q^{2-n_{13}+s}\Big(q^{-n_{22}-n_{31}}[n_{13}][n_{22}]t_{11}^{n_{11}}t_{12}^{n_{12}+1}t_{13}^{n_{13}-1}t_{21}^{n_{21}+1}t_{22}^{n_{22}-1}\dots t_{33}^{n_{33}}\\&+q^{-n_{22}-n_{32}}[n_{13}][n_{32}]t_{11}^{n_{11}}t_{12}^{n_{12}+1}t_{13}^{n_{13}-1}\dots t_{31}^{n_{13}+1}t_{32}^{n_{32}-1}t_{33}^{n_{33}}\\&+q^{-n_{23}-n_{32}}[n_{23}][n_{32}]t_{11}^{n_{11}}\dots t_{22}^{n_{22}+1}t_{23}^{n_{23}-1}t_{31}^{n_{31}+1}t_{32}^{n_{32}-1}t_{33}^{n_{33}}\Big).\end{aligned} \tag{A18}$$

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